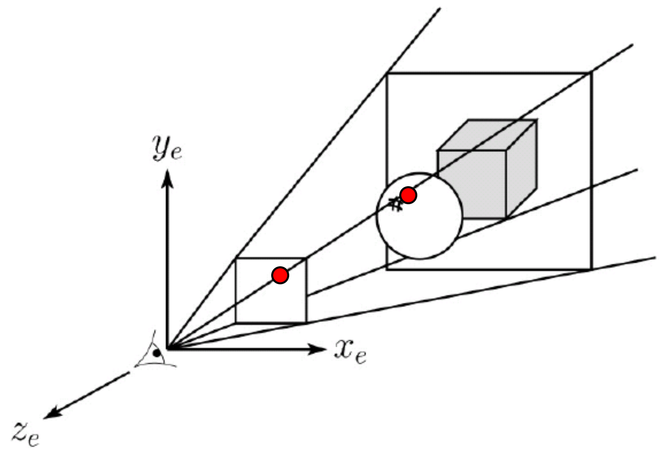
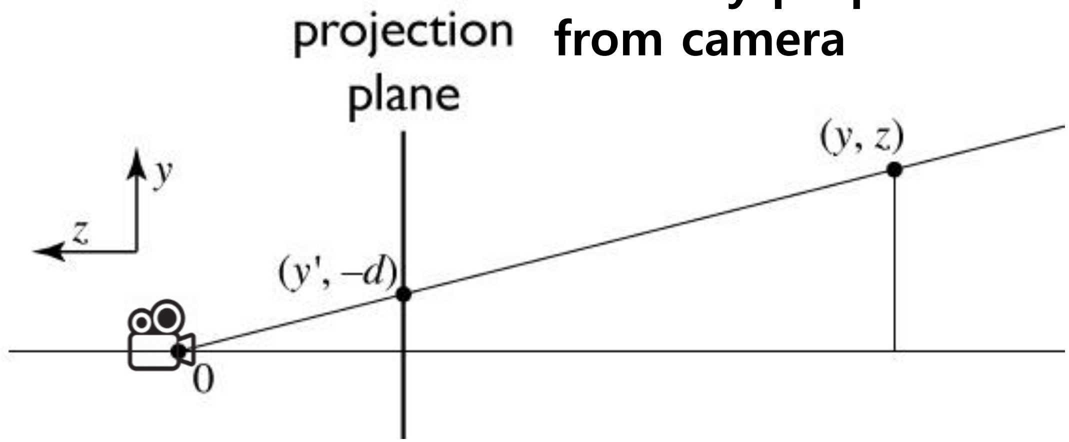
Let's first consider 3D View Frustum→2D Projection Plane

- Consider the projection of a 3D point on the camera plane



Perspective projection

The size of an object on the screen is inversely proportional to its distance from camera



similar triangles:

$$\frac{y'}{d} = \frac{y}{-z}$$

$$y' = -dy/z$$

Homogeneous coordinates revisited

- Perspective requires division
  - that is not part of affine transformations
  - in affine, parallel lines stay parallel
    - therefore not vanishing point
    - therefore no rays converging on viewpoint
- "True" purpose of homogeneous coords: projection

Homogeneous coordinates revisited

- Introduced *w* = 1 coordinate as a placeholder

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

  – used as a convenience for unifying translation with linear

- Can also allow arbitrary *w*

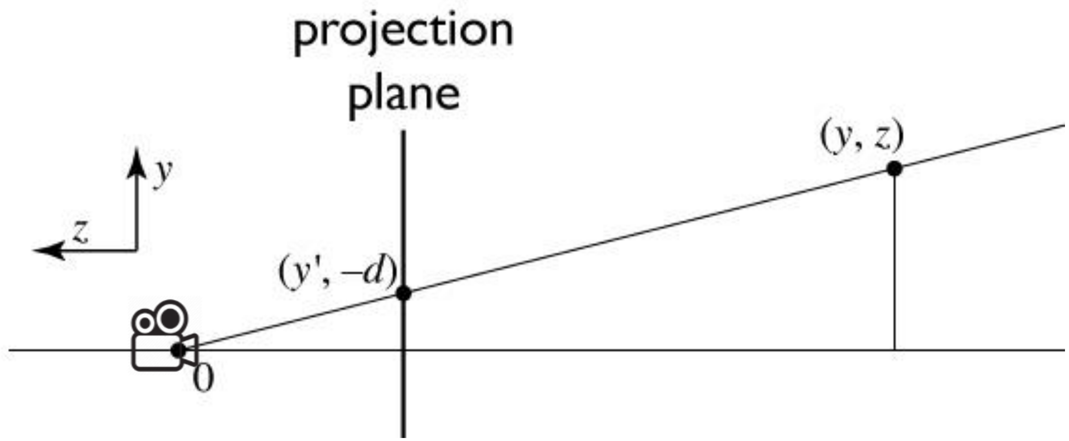
$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \sim \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

Implications of *w*

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \sim \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

- All scalar multiples of a 4-vector are equivalent
- When *w* is not zero, can divide by *w*
  - therefore these points represent “normal” affine points
- When *w* is zero, it’s a point at infinity, a.k.a. a direction
  - this is the point where parallel lines intersect
  - can also think of it as the vanishing point
- Digression on projective space

Perspective projection

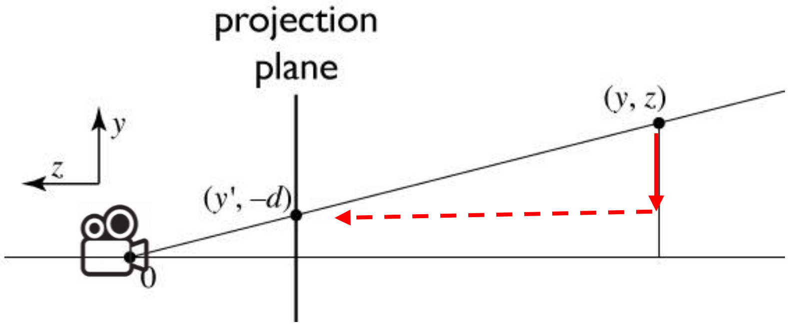


to implement perspective, just move z to w:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

So far, 3D View Frustum→2D Projection Plane

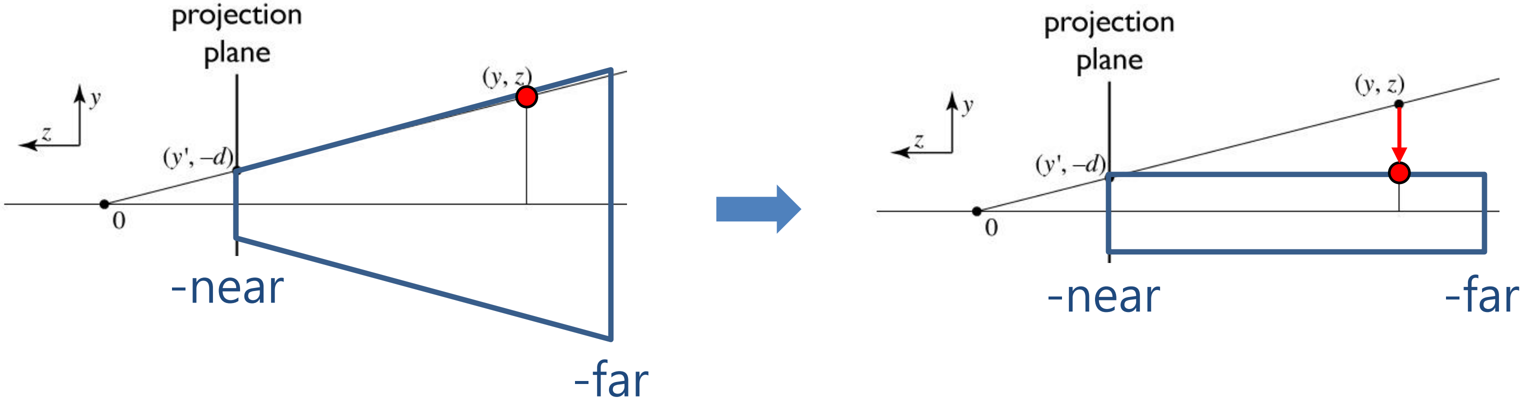
- What we've just seen is a story of 3D → 2D



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Now, 3D View Frustum→3D Cuboid

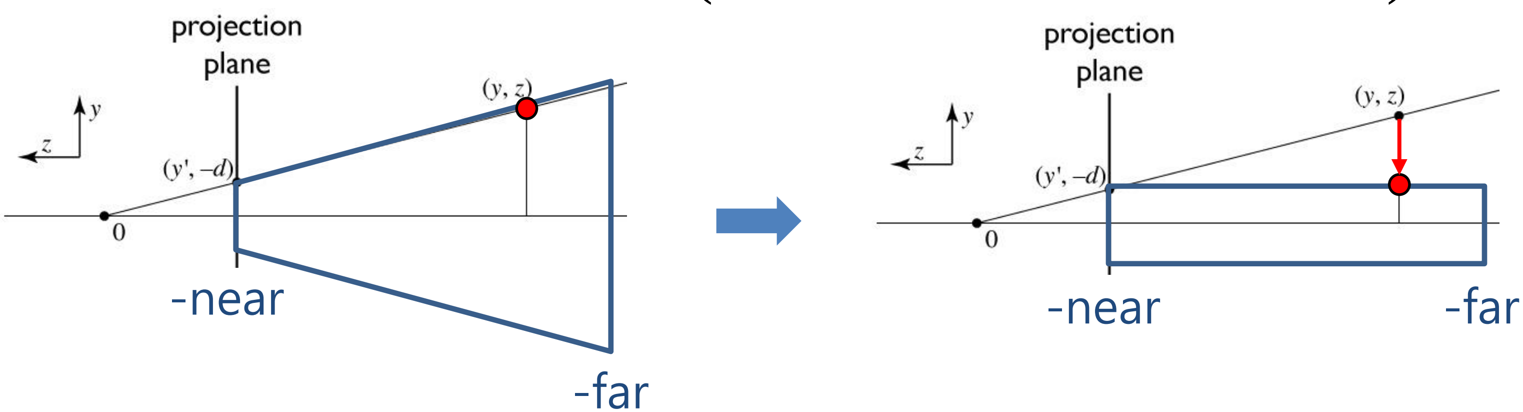
- What we have to do is 3D → 3D
  - Let's first consider a viewing frustum → a cuboid with the same near and far offset (not a canonical view volume)



- Perspective has a varying denominator—can't preserve depth z!
- Instead, preserve depth on near and far planes.

Now, 3D View Frustum→3D Cuboid

- What we have to do is 3D → 3D
  - Let's first consider a viewing frustum → a cuboid with the same near and far offset (not a canonical view volume)

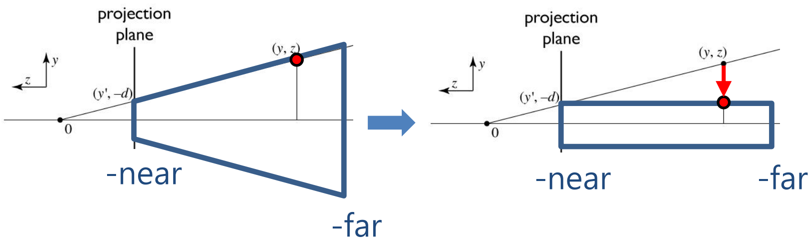


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ -\ X\ /z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ X \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ c_0 & c_1 & a & b \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Perspective has a varying denominator—can't preserve depth z!

3D View Frustum→3D Cuboid

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ -X/z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ X \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ c_0 & c_1 & a & b \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



- Note that z' is independent of x and y, therefore $c_0 = c_1 = 0$
- We want z depth -near → -near, -far → -far

$$X = az + b \qquad z'(z) = -X/z = -\frac{az+b}{z}$$

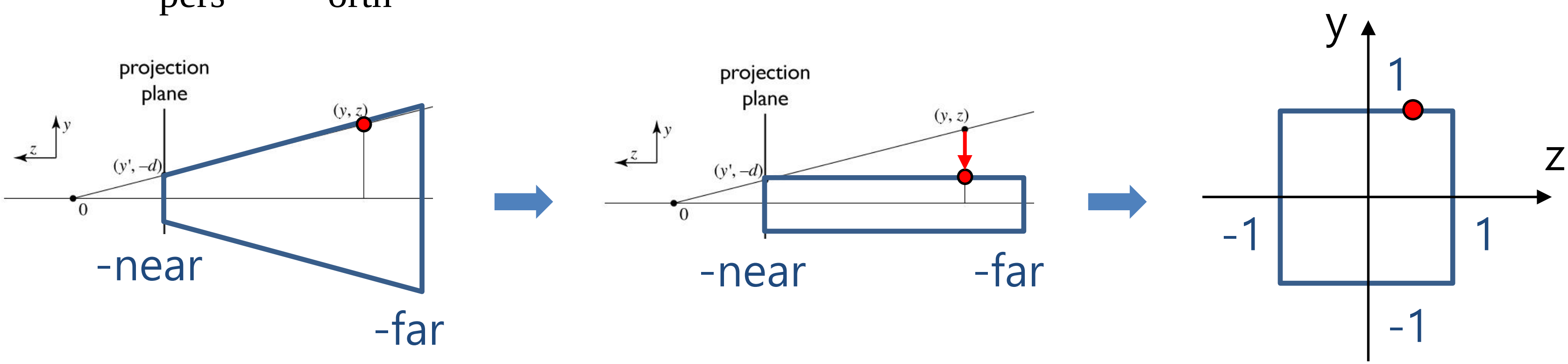
- Find 2 unknowns a, b with 2 eq. z'(-n)=-n, z'(-f)=-f
- → a = f+n, b = fn (try it)

Final: 3D View Frustum→3D Canonical View Volume

- By substituting d with n, P= $\begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & f+n & fn \\ 0 & 0 & -1 & 0 \end{bmatrix}$

- Now the remaining work is mapping the cuboid to a canonical view volume: M_{orth}

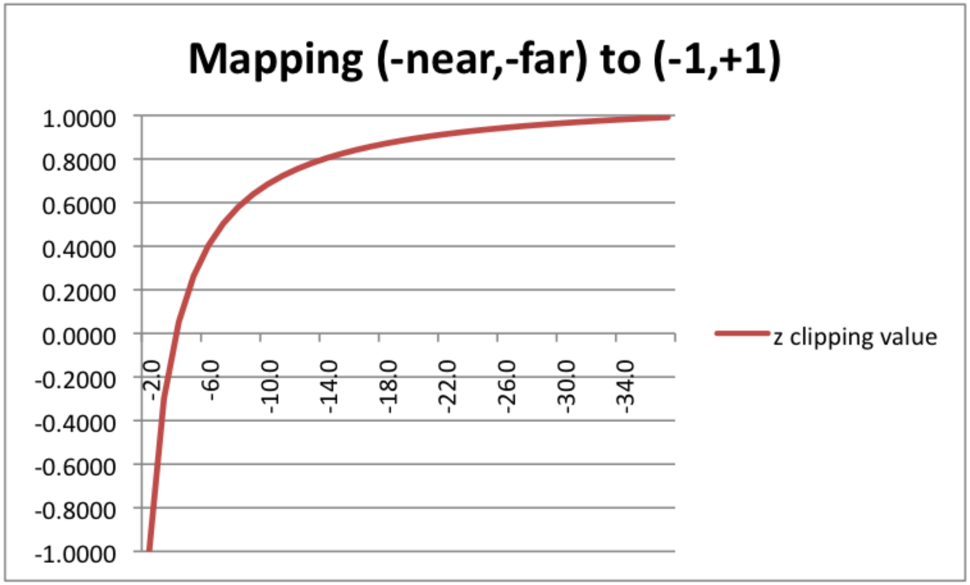
- Viewing frustum → cuboid → canonical view volume: $M_{pers} = M_{orth}\, P$



Final: 3D View Frustum→3D Canonical View Volume

- This projection matrix M_{pers} results in non-linear mapping of the original depth values (z values) to the range (-1, +1).

- This makes more precision for z values close to the camera and less precision for z values farther from the camera.



Perspective Projection Matrix

- $M_{pers} = M_{orth}\, P$

$$= \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & f+n & fn \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Acknowledgement

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  - Prof. Taesoo Kwon, Hanyang Univ., http://calab.hanyang.ac.kr/cgi-bin/cg.cgi
  - Prof. Steve Marschner, Cornell Univ., http://www.cs.cornell.edu/courses/cs4620/2014fa/index.shtml